ON DIAGRAM-CHASING IN DOUBLE COMPLEXES

by George M. Bergman*

Introduction. The "magic" of diagram-chasing consists in establishing relations between distant points of a diagram — exactness-implications, connecting morphisms, etc.. These "long" connections are in general composites of "short" (unmagical) connections, but the latter, and even the objects they join, are frequently not visible in the diagram-chasing proof. We attempt to remedy this situation here.

Given a double complex in an abelian category, we consider for each object A of the complex the horizontal and vertical homology objects at A, and two other objects, denoted ${}^{\square}A$ and $A_{\square}$. For each arrow of the double complex we construct a 6-term exact sequence of these objects. Standard results such as the 3x3-Lemma, the Snake Lemma and the long exact sequence of homologies associated with a short exact sequence of complexes are shown to be easy applications of this exact sequence.

We then develop some further (rather baroque) results along the lines of the last mentioned application, obtaining various diagrams of exact sequences from complexes with almost all rows and columns assumed exact.

The total homology of a double complex is also examined in terms of the objects ${}^{\square}A$ and $A_{\square}$.

For curiosity we take a brief look at the more complicated world of triple complexes.

The relation between the ideas developed here and J. Lambek's homological formulation of Goursat's Theorem [3] is examined in §6.3.

We end with some exercises.

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1. Basics.

We shall work in an abelian category $\underset{\sim}{A}$. Capital letters and points in diagrams will represent objects of $\underset{\sim}{A}$; lower-case letters and arrows in diagrams will represent morphisms in $\underset{\sim}{A}$.

A double complex will mean an array of objects and maps in $\underset{\sim}{A}$ as in Figure 1, in general extending indefinitely in both directions, in which every row and every column is a complex (successive arrows compose to zero) and all squares commute.* Note that a "partial double complex" such as Fig. 2 (the diagram for the 4-lemma, consisting of two 4-term exact sequences and morphisms making commuting squares) can be made a double complex by completing it with 0's on all sides; or by writing in some kernels and cokernels, and then 0's. Thus our results on double complexes will be applicable to such diagrams.

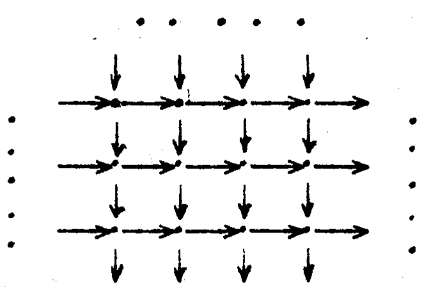

Fig. 1

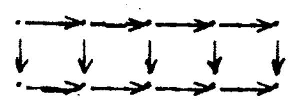

Fig. 2

Topologists often prefer double complexes with <u>anti</u>commuting squares, but it is a familiar observation that one type of complex can be turned into the other by reversing the signs of the arrows in every other row (or column). In the theory of spectral sequences the vertical arrows of a double complex are generally drawn going upward, while in results like the 4-lemma they are drawn downward; I shall follow the latter convention.

*The reader should note that in diagrams where some objects are represented by letters and others by dots, the latter are <u>not</u> assumed zero; they are simply objects we shall not need to refer to by name.

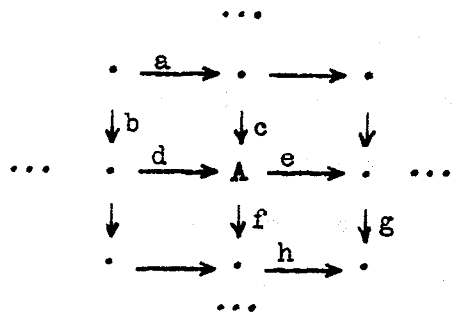

Figure 3.

<u>Definition 1.</u> Let A be an object of a double complex, and let maps be labeled as in Figure 3. Then we define:

$A{\leftarrow}$ = Ker e / Im d, the horizontal homology object at A,

$A{\updownarrow}$ = Ker f / Im c, the vertical homology object at A,*

$^{\square}A$ = (Ker e ∩ Ker f)/Im p (where p = ca = db), which we shall call the <u>receptor</u> at A,

$A_{\square}$ = Ker q / (Im c + Im d) (where q = ge = hf), which we shall call the <u>donor</u> at A.

From the inclusion relations among the kernels and images in Definition 1 we get

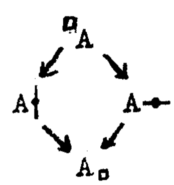

Figure 4.

<u>Lemma-Definition 2.</u> For any object A of a double complex, the identity map of A induces a commuting diagram of maps as shown in Figure 4.

We shall call these the <u>intermural maps</u> associated with the object A.

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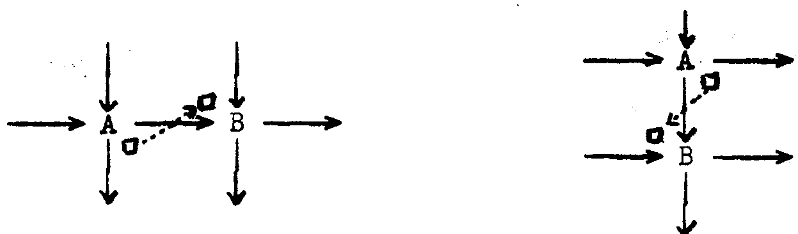
*Actually, I would have liked <A> and $\hat{\check{A}}$ for the horizontal and vertical homology objects — they make the results to come more distinctive to the eye. But $\hat{\check{A}}$ is probably typographically unfeasable. A possible alternative would be <A> and A$\hat{\ }$.

In the diagram for a double complex, the donor and receptor at a given object will generally be indicated by small squares to the lower-right and upper-left of the point or letter representing that object. (Cf. Figure 5 and later figures.) Note that the side toward which the square is displaced corresponds to the direction of the most distant object involved in the definition of the quotient-object in question, (the domain or codomain of the composite arrow p or q). If one draws a double complex with vertical arrows going up, one should of course write ${}_{\square}A$ and $A^{\square}$ for the receptor and donor at **A.** Occasionally we will indicate horizontal and vertical homology objects by marks $\rightarrow$ and $\vert$ placed over the location of the object, but this requires suppressing the symbol for the object itself.

The motivation for the names "donor" and "receptor" is seen in:



(a) (b)

Figure 5.

Lemma-Definition 3. Each arrow $f: A \rightarrow B$ in a double complex induces an arrow $A_{\square} \rightarrow {}^{\square}B$, which we shall call the extramural map associated with f.

The global picture of the extramural maps in a double complex is shown in Figure 6.

I shall not introduce any special symbols for these maps; others may wish to do so, but since between any two of the objects we have constructed we do not define more than one map, we shall be able to get by in this note with unlabeled arrows, representing the unique maps defined between the objects named.

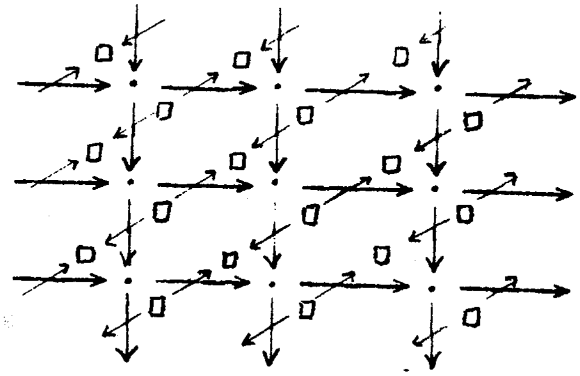


Figure 6.

To show a <u>composition</u> of the maps we have defined, we shall use a single arrow marked with dots indicating the intermediate objects involved, as in the statement of the next result.

<u>Lemma 4.</u> If $f: A \to B$ is a <u>horizontal</u> arrow of a double complex (Figure 5(a)) then the (well known) induced map of <u>vertical</u> homology objects, $A\nmid \to B\nmid$, is given by the composition of two intermural and one extramural maps:

$$A\nmid \xrightarrow{\quad A_{\square} \bullet \quad {}^{\square}B \bullet \quad} B\nmid .$$

Likewise, for a <u>vertical</u> map $A \to B$ (Figure 5(b)) the induced map of <u>horizontal</u> homology objects is given by

$$A\!+ \xrightarrow{\quad A_{\square} \bullet \quad {}^{\square}B \bullet \quad} B\!+ .$$

We now come to our main Lemma. We will again state both the horizontal and vertical cases, since we will have numerous occasions to use both. The verifications are trivial if one is allowed to look at elements; one may apply the standard tricks for translating such a proof into one working in a general abelian category (cf. [6], Theorem 3.3)* or call on an embedding theorem (see references in [6], p.205). [illegible] by analogy [illegible] the "Snake Lemma".

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*Part (vi) of that Theorem might be replaced by the following more convenient form: "If... $gx \equiv gy$, then there exist $x' \equiv x$, $y' \equiv y$ [illegible] the same domain, such that $gx'=gy'$.

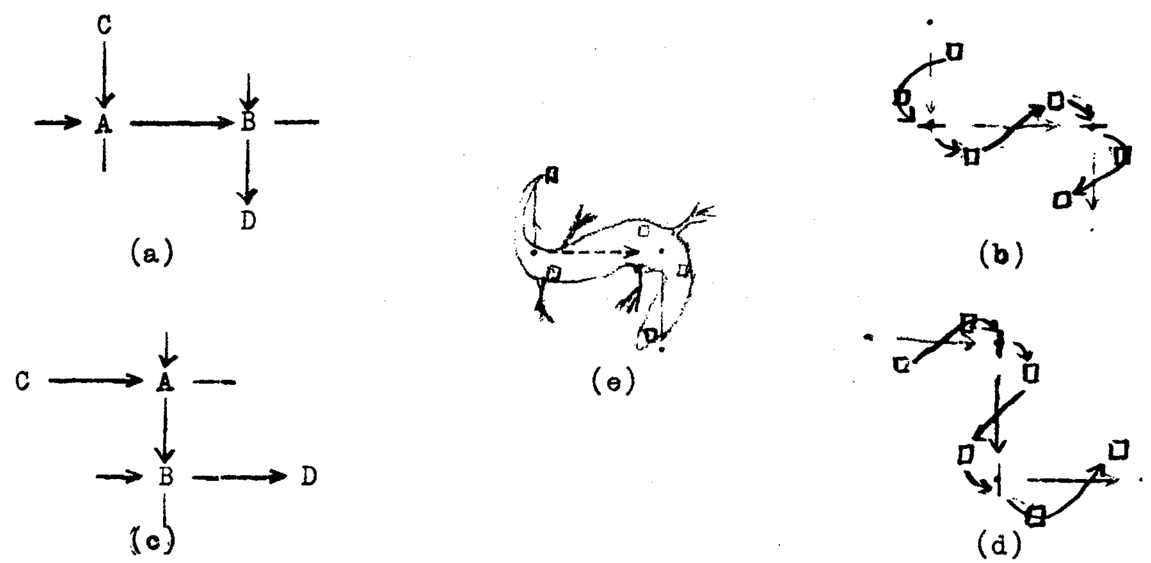

Figure 7.

<u>Lemma 5 (Salamander Lemma)</u>. Let $A \longrightarrow B$ be a horizontal arrow in a double complex, and C, D the objects above A and below B, respectively, as in Figure 7(a). Then the following sequence formed from intramural and extramural maps (Figure 7(b)) is exact:

$$C_{\square} \xrightarrow{{}^{\square}A} A{\leftarrow} \longrightarrow A_{\square} \longrightarrow {}^{\square}B \longrightarrow B{\leftarrow} \xrightarrow{B_{\square}} {}^{\square}D$$

Likewise, if $A \longrightarrow B$ is a vertical arrow, we have the exact sequence

$$C_{\square} \xrightarrow{{}^{\square}A} A{\downarrow} \longrightarrow A_{\square} \longrightarrow {}^{\square}B \longrightarrow B{\downarrow} \xrightarrow{B_{\square}} {}^{\square}D.$$

In either case, we shall call the sequence displayed "the 6-term exact sequence associated with the map $A \longrightarrow B$ of the given double complex."

2. Degenerate cases and easy applications.

Note that the extramural arrows in Figure 6 stand head-to-head and tail-to-tail, and so cannot be composed. But this difficulty is clearly removed under appropriate conditions by the following result, which we prove by taking the two homology objects in the 6-term exact sequence(s) of Lemma 5 to be 0:

<u>Corollary 6.</u> Let $A \to B$ be an arrow in a double complex, and suppose the row or column containing this map is exact at both A and B. Then the induced extramural map is an isomorphism: $A_{\square} \xrightarrow{\sim} {}^{\square}B$.

Using another degenerate case of Lemma 5 we can relate our donors and receptors to the conventional homology objects:

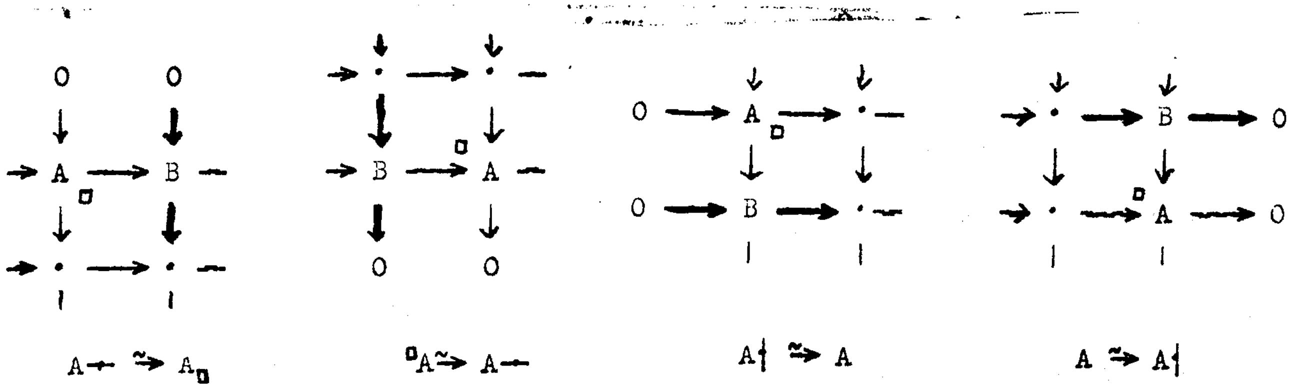

Figure 8

<u>Corollary 7.</u> In each of the situations of Figure 8, if the darkened row or column is exact at B, then the intermural map at A indicated below the diagram is an isomorphism.

<u>Proof</u> (for the first case; others are equivalent.) By Corollary 6 applied to the arrow $0 \to B$ we get ${}^{\square}B \cong 0_{\square} = 0$. So the 6-term exact sequence associated with the map $A \to B$ begins $0 \to A\!\leftarrow \to A_{\square} \to 0 \to \ldots$, giving the desired isomorphism. (The sequence associated with the map $0 \to A$ now likewise gives $A\!\downarrow \cong {}^{\square}A$, and analogously in the remaining three cases, but we will not need this!)

Most of the standard "small" diagram-chasing lemmas of homological algebra can be obtained from the special cases of Lemma 5 given by the above two Corollaries. For example:

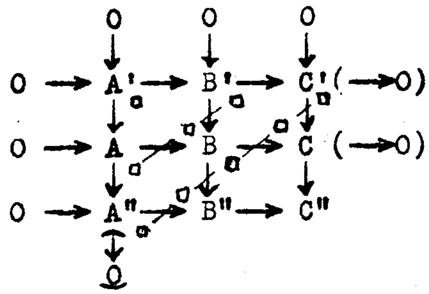

Figure 9.

Lemma 8 (The Sharp 3x3 Lemma, [2] p.365) If in Figure 9, ignoring the parenthesized arrows, all columns and all but the first row are exact, then the first row is also exact.

If with the parenthesized arrows added the first column and middle row are still exact, then again so is the first row.

Proof. Using the exactness assumptions, and the preceding corollaries, we get:

$$A'\!\leftarrow \;\cong\; A'_{\square} \cong A'\!\downarrow = 0,$$

$$B'\!\leftarrow \;\cong\; B'_{\square} \cong {}^{\square}B \cong A_{\square} \cong A\!\downarrow = 0,$$

and under the stronger hypotheses,

$$C'\!\leftarrow \;\cong\; C'_{\square} \cong {}^{\square}C \cong B_{\square} \cong {}^{\square}B'' \cong A''_{\square} \cong A''\!\downarrow = 0.$$

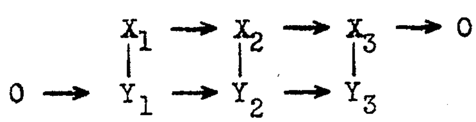

Figure 10

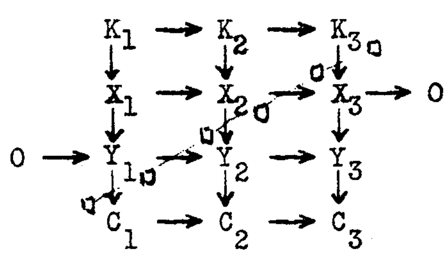

Figure 11

Lemma 9 (Snake Lemma, [2] p.50). If in Figure 10 both rows are exact, and if we add in a row of kernels and a row of cokernels as in Figure 11, then these two rows fit together in an exact sequence

(1) $$K_1 \to K_2 \to K_3 \to C_1 \to C_2 \to C_3.$$

Proof. We extend Figure 11 to a double complex by adding in the kernel X_0 of the map $X_1 \to X_2$, the cokernel Y_4 of the map $Y_2 \to Y_3$, and zero's everywhere else. Thus, the middle three columns and the middle two rows are exact.

The exactness of (1) at K_2 now follows from the isomorphisms

$$K_2^{\leftarrow} \cong K_{2\square} \cong {}^{\square}X_2 \cong X_{1\square} \cong {}^{\square}Y_1 \cong 0_\square = 0,$$

and exactness at C_2 is shown similarly.

We now want to get a connecting map $K_3 \to C_1$ making (1) exact at these two objects. This is equivalent to getting an isomorphism between $\mathrm{Cok}(K_2 \to K_3)$ and $\mathrm{Ker}(C_1 \to C_2)$. But indeed, such an isomorphism is given by the following composition of two intermural and five extramural maps:

$$\mathrm{Cok}(K_2 \to K_3) = K_3^{\leftarrow} \cong K_{3\square} \cong {}^{\square}X_3 \cong X_{2\square} \cong {}^{\square}Y_2 \cong Y_{1\square} \cong {}^{\square}C_1 \cong C_1^{\leftarrow} = \mathrm{Ker}(C_1 \to C_2).$$

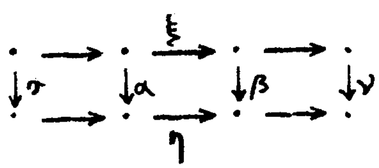

Figure 12.

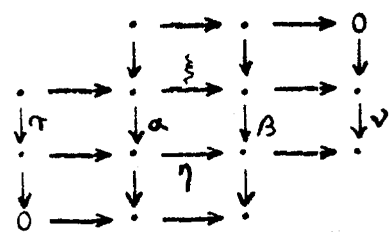

Figure 13.

Lemma 10. (Strong Four Lemma [2], p.14). If in the commutative diagram of Figure 12 the rows are exact, ν is a monomorphism, and τ an epimorphism, then $\xi(\mathrm{Ker}\ \alpha) = \mathrm{Ker}\ \beta$, and $\mathrm{Im}\ \alpha = \eta^{-1}(\mathrm{Im}\ \beta)$.

Idea of proof: Add in the vertical kernels and cokernels shown in Figure 13. and zeroes everywhere else, and note that the result is a double complex. The desired conclusion is equivalent to the exactness of the two new rows, which one verifies "as usual".

The Five Lemma ([2], p. 14) is an immediate consequence of the above result.

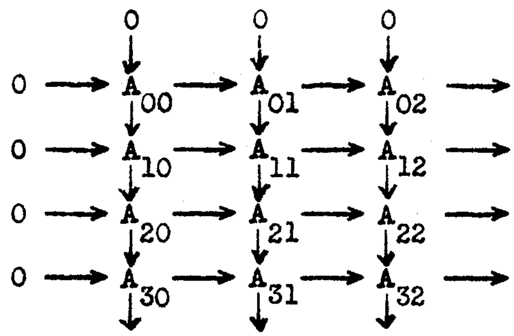

Figure 14.

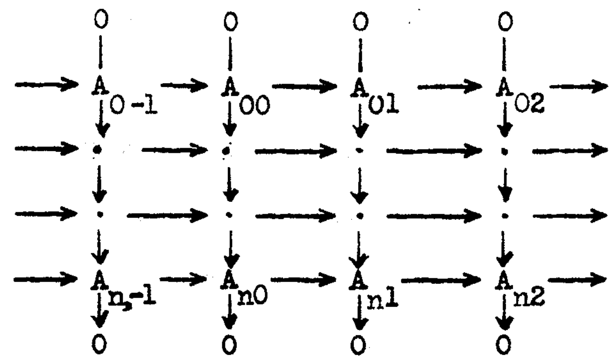

Figure 15.

Lemma 11. If in Figure 14* all rows but the first, A_{0r}, and all columns but the first, A_{r0}, are exact, then the homologies of the first row and the first column are isomorphic: $A_{0r}\rightarrow \cong A_{r0}\downarrow$. (And analogously for a double complex bordered on the bottom and right by zeroes.)

If in Figure 15 all columns are exact, and all rows but the first and last are exact, then the homologies of these two rows agree except for a shift by n-1: $A_{0r}\rightarrow \cong A_{n,1-n+r}\rightarrow$. (And analogously for a vertical array bordered on right and left by zeroes.)

(On the other hand, if we form a double complex as in Figure 14 but with zeroes on the top and right, or on the bottom and left, and again, all but the edge row and column exact, there need be no relation between the homologies of these.)

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*By an arbitrary choice, when objects of a double complex are to be distinguished by numerical subscripts, I shall number it as a double cochain complex. If readers of this preprint tell me that for some reason the reverse choice is preferable, I will change the final version.

2. **Weakly bounded double complexes.**

Before exploring the uses of the full statement of Lemma 5, it will be useful to consider a mild generalization of our last result.

Suppose as in Figure 15 that we have a double complex with exact columns, bounded above and below by rows of 0's. Rather than assuming exactness in all but the top and bottom rows, let us assume it in all rows but the i^{th} and j^{th}, for arbitrary i and j with $0 \leq i < j \leq n$. I claim it will still be true that the homologies of these rows agree up to a shift:

(2) $$A_{i,r}^{\leftarrow} \cong A_{j,r-(j-i+1)}^{\rightarrow}$$

Indeed, by composing extramural isomorphisms we see:

(3) $$A_{i,r\,\square} \cong {}^{\square}A_{j,r-(j-i-1)};$$

the problem is to strengthen Corollary 7 to show that these objects are isomorphic (by the intermural maps) to the homology objects of (2). But if one examines the proof of that Corollary, one sees that all that is used is the vanishing of certain donor and receptor objects near A:

$^{\square}A \cong A^{\downarrow}$ $\qquad$ $^{\square}A \cong A^{\leftarrow}$

$A_{\square} \cong A^{\leftarrow}$ $\qquad$ $A_{\square} \cong A^{\downarrow}$

(a) $\qquad$ (b)

Figure 16.

Corollary 12 (to proof of Corollary 7). If A is an object of a double complex, as in Figure 16 (a) or (b), and the adjacent donor and receptor objects marked "o" in that figure are zero, then the two intermural isomorphisms indicated below that figure hold.

Now in the situation we are interested in, our complex is exact horizontally and vertically above the i^{th} row, so we can use Corollary 6 to connect any donor or receptor above the i^{th} row to 0. Hence we can use the above Corollary to get the isomorphism between the left-hand sides of (2) and (3). Similarly, using exactness below the j^{th} row we can connect the right-hand sides, and thus we get (3) from (2), as claimed.

What if we have a double complex in which all columns and all but the i^{th} and j^{th} rows are exact, but we do not assume that all but finitely many rows are zero? Starting from the receptor at any object in the i^{th} row, we can still get an infinite chain of isomorphisms going diagonally upward,

$$^{\square}A_{i,s} \cong A_{i-1,s\,\square} \cong {}^{\square}A_{i-1,s+1\,\square} \cong {}^{\square}A_{i-2,s+1} \cong \ldots,$$

but we can no longer conclude that the common value is zero; and similarly below the j^{th} row. However, there are certainly other hypotheses than the one we were using above that <u>will</u> allow us to say this common value is 0; e.g., the existence of zero quadrants (rather than half-planes) to the upper right and lower left. Let us make, generally,

<u>Definition 13.</u> A double complex with objects $A_{r,s}$ will be called <u>weakly bounded</u> if for every r and s, there exists a <u>positive</u> integer n such that ${}^{\square}A_{r-n,s+n}$ or $A_{r-n-1,s+n\,\square}$ is zero, and also a <u>negative</u> integer n with the same property.

The above discussion now immediately yields the first statement of the next Corollary; the second statement follows from a similar argument.

<u>Corollary 14</u> (to proof of Lemma 11). Let us be given a <u>weakly bounded</u> double complex, with objects $A_{r,s}$.

If all columns are exact, and all rows but the i^{th} and j^{th}, with $i < j$, then the homologies of these rows are isomorphic except for a shift: $A_{i,r}{}^{-\bullet-} \cong A_{j,r-(j-i-1)}{}^{-\bullet-}$. (And analogously if all but two columns are exact.)

If all rows but the i^{th} and all columns but the j^{th} are exact (i and j arbitrary), then the i^{th} row and j^{th} column have isomorphic homologies: $A_{i,r} \cong A_{j,r-j+i}$.

To see that the above Corollary fails without the hypothesis of weak boundedness, consider Figure 17, where A is any nonzero object. All rows and all columns are exact except the row with just a single "A". This "contradicts" both parts of the Corollary.

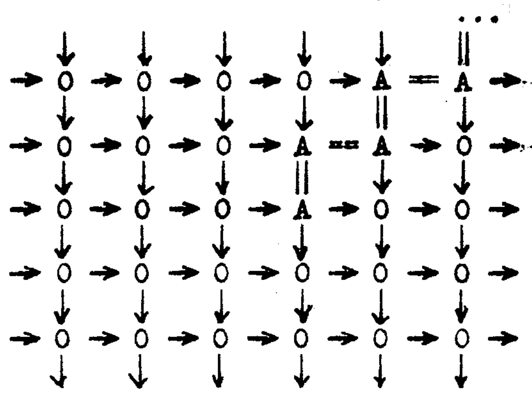

Figure 17.

3. Long exact sequences.

At this point it would be easy to apply Lemma 5 (and Corollaries 6 and 7) to give a quick construction of the long exact sequence of homologies associated with a short exact sequence of complexes; the reader may stop and do so himself. But we shall find it more instructive to examine how the six-term exact sequences associated with a double complex link together under various weak hypotheses, and see that the above long exact sequence is the simplest interesting case of some more general phenomena.

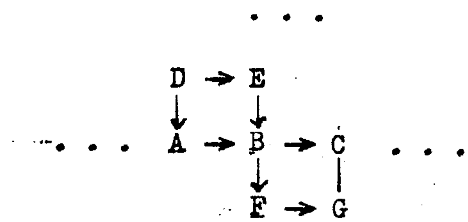

Figure 18.

If B is any object of a double complex (Figure 18) let us consider the 6-term exact sequences associated with the four arrows into and out of B. These are shown together in Figure 19* (in which the central square and all four "triangles" commute!)

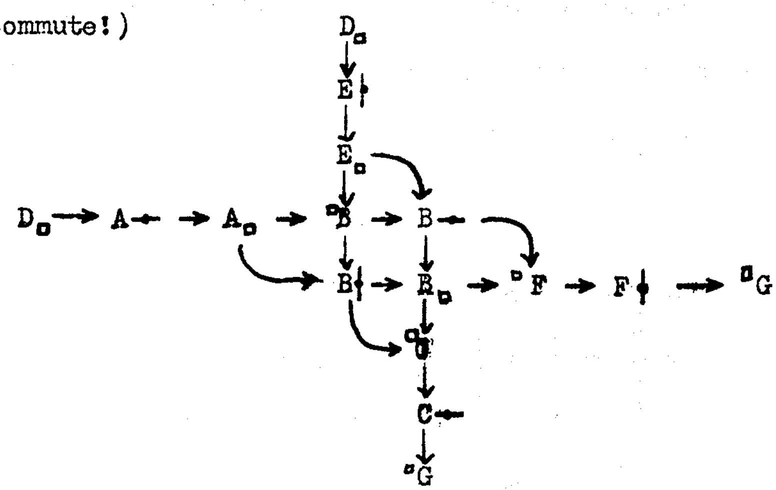

Figure 19.

Now note what happens if our given double complex (Figure 18) is vertically exact <u>at</u> B.

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*One would like to be able to piece together copies of Figure 19 to get a nice global diagram of the exact sequences associated with all the arrows of a double complex. But note that in Figure 19, the cyclic order in which the four sequences are drawn is not the same as that of the maps in Figure 18 to which they correspond. This frustrates the hoped-for "piecing together". In fact, any attempt at a "global diagram" without additional hypotheses ends up looking like a trayful of squirming salamanders.

Lemma 15. Suppose in the situation of Figure 18 that $B{\downarrow} = 0$ (or more generally, that either of the maps $^{\square}B \to B{\downarrow}$ or $B{\downarrow} \to B_{\square}$ is zero.) Then the following sequence (obtained from the "left-hand" and "bottom" branches of Figure 19, which dorrespond to the maps $A \to B \to C$ of Figure 18) is exact:

(4) $$D_{\square} \to A{\leftarrow} \to A_{\square} \to {}^{\square}B \to B{\leftarrow} \to B_{\square} \to {}^{\square}C \to C{\leftarrow} \to {}^{\square}G$$

(Likewise, if $B{\leftarrow} = 0$, we get an exact sequence from the other two branches of Figure 19.)

Proof. We know that the 6-term sequences of which Figure 19 is composed are all exact; this gives us the exactness of (4) everywhere except at $B{\leftarrow}$. There we know (from the "bottom" branch) that $(\mathrm{Ker}\ B{\leftarrow} \to B_{\square}) = \mathrm{Im}(E_{\square} \to B{\leftarrow})$. But if the arrow $^{\square}B \to B{\downarrow}$ is zero, then by exactness of the "top" branch, $E_{\square} \to {}^{\square}B$ is onto, so $\mathrm{Im}({}^{\square}B \to B{\leftarrow}) = \mathrm{Im}(E_{\square} \to B{\leftarrow}) = \mathrm{Ker}(B{\leftarrow} \to B\ )$, proving the exactness of (4). The corresponding argument gives the same result if the map $B{\downarrow} \to B_{\square}$ is zero.

Corollary 16. If in a double complex as in Figure 18 the vertical homologies are zero for all objects in the row $\ldots \to A \to B \to C \to \ldots$, then we get a long exact sequence of associated objects and intermural and extramural maps:

$$\ldots \to {}^{\square}A \to A{\leftarrow} \to A_{\square} \to {}^{\square}B \to B{\leftarrow} \to B_{\square} \to {}^{\square}C \to C{\leftarrow} \to C_{\square} \to \ldots$$

Corollary 17. If in a double complex (as in Figure 20) all columns are exact, then the rows induce a system of long exact sequences linked by isomorphisms, as in Figure 21.

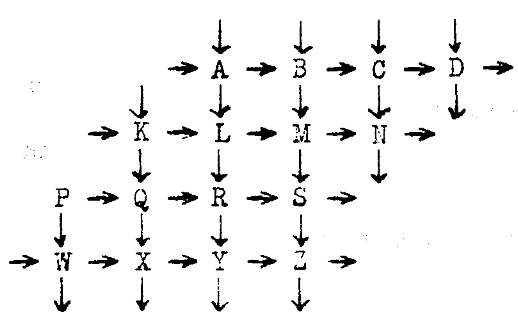


Figure 20.

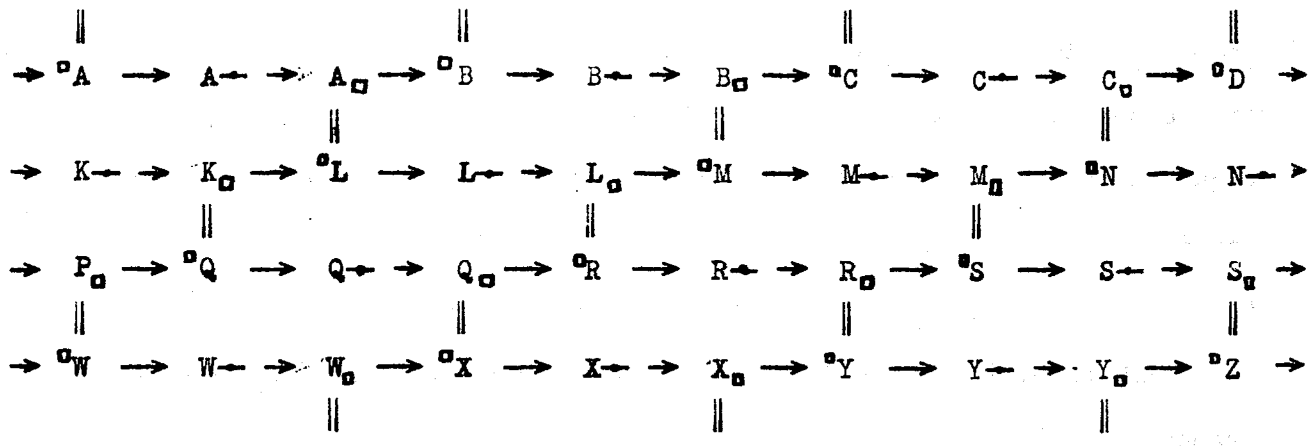


Figure 21.

Note that in these long exact sequences the homology objects form every third term — the terms of Figure 21 that are not connected either above or below by isomorphisms.

Suppose now that in our original double complex, in addition to all columns being exact, some row is exact. This means that in the induced system of long exact sequences, the corresponding row will have every third term zero; and so the maps connecting the remaining terms will be isomorphisms (Figure 22).

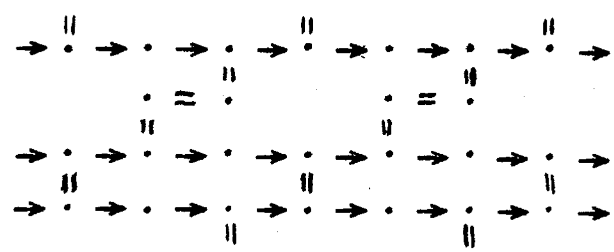


Figure 22.

We see that these, together with the vertical isomorphisms, tie together the preceding and following exact sequence to give a system essentially like Figure 21 again, except for a horizontal shift by one period. If n successive rows of the double complex are exact, we get a similar "annealing" with a shift by n periods.

If all rows are exact above a certain point, then we get infinite chains of isomorphisms going upward and to the right. If the complex is also weakly bounded (Definition 13) the common values along these chains will be zero; in particular, every third term of the long exact sequence corresponding to the top nonexact row will be zero, and again we get isomorphisms between successive remaining terms. This time, half of these will be horizontal homology objects. If we consider homology objects "more important" than donors and receptors, we will use these isomorphisms and the isomorphisms joining this row to the one below to insert these objects in that row in place of all the receptors. Concretely, suppose in Figure 20 that all rows above the top one shown are exact, and the complex is weakly bounded above. Then the top two rows of Figure 21 will take the form

$$\begin{array}{ccccccccc} & & & & A^{\leftarrow} = & A_{\square} & & & \\ & & & & & \| & & & \\ \to & K^{\leftarrow} & \to & K_{\square} & \to & {}^{\square}L & \to & L^{\leftarrow} & \to \dots \\ & & & \| & & & & & \end{array}$$

which we rewrite

$$\begin{array}{ccccccccc} \to & K^{\leftarrow} & \to & K_{\square} & \to & A^{\leftarrow} & \to & L^{\leftarrow} & \to \dots \\ & & & \| & & & & & \end{array}$$

Thus we get a system of long exact sequences in which the top sequence has homology objects for two out of every three terms; and the same for the bottom sequence if all terms below some point are exact. When there are only three nonexact rows, we get a single sequence with all terms homology objects:

Lemma 18. Suppose we are given a weakly bounded double cochain complex, with objects A_{rs}, all columns exact, and all rows exact except the i^{th}, j^{th} and k^{th}, where $i < j < k$. Let $m = j-i-1$, $n = k-j-1$. Then we get a long exact sequence

$$\dots \to A_{i,r+m}{}^{\leftarrow} \to A_{j,r}{}^{-} \to A_{k,r-n}{}^{\bullet\to} \to A_{i,r+m+1}{}^{\leftarrow} \to A_{j,r+1}{}^{-} \to A_{k,r-n+1}{}^{\bullet\to} \to \dots$$

For comparison, we note that if in Figure 20, all rows but the four rows shown are exact (in addition to all columns), and the complex is weakly bounded, then Figure 21 (precisely: its middle two rows) become Figure 23:

$$\begin{array}{cccccccccccccccccccc}
\to & K_\leftarrow & \to & K_\square & \to & A_\leftarrow & \to & L_\leftarrow & \to & L_\square & \to & B_\leftarrow & \to & M_\leftarrow & \to & M_\square & \to & C_\leftarrow & \to & N_\leftarrow \to \\
 & & & \| & & & & & & \| & & & & & & \| & & & & \\
\to & W_\leftarrow & \to & {}^\square Q & \to & Q_\leftarrow & \to & X_\leftarrow & \to & {}^\square R & \to & R_\leftarrow & \to & Y_\leftarrow & \to & {}^\square S & \to & S_\leftarrow & \to & Z_\leftarrow \to
\end{array}$$

Figure 23.

One piece of unfinished business — Figure 19. In Lemma 15, we saw how the 6-term exact sequences corresponding to successive arrows of Figure 18, $A \to B \to C$ (or $E \to B \to F$) can link together in a 9-term sequence. Under other hypotheses, the sequences corresponding to a pair of arrows that go "around a corner", such as $E \to B \to C$ (or $A \to B \to F$) link to give an 8-term exact sequence. (I formulate the result, not so much with the idea that it should be used and quoted, as that it should be suggestive of how to work with 6-term sequences.)

<u>Lemma 19.</u> In the situation of Figure 18, if

(a) $E_\square = 0$ (or more generally if the map $E_\square \to {}^\square B$ is zero), resp. if

(b) the maps $E_\square \to B_\leftarrow$ and $B_\upharpoonleft \to {}^\square C$ are zero, resp. if

(c) ${}^\square C = 0$ (or more generally if the map $B_\square \to {}^\square C$ is zero),

then the left- and right-hand branches of Figure 19 combine to give an exact sequence; namely, according to the respective case:

$$D_\square \to A_\leftarrow \to A_\square \left\{ \begin{array}{ccccc} \to & B_\upharpoonleft & \to & B_\square & \to \\ \to & {}^\square B & \to & B_\square & \to \\ \to & {}^\square B & \to & B_\leftarrow & \to \end{array} \right\} {}^\square F \to F_\upharpoonleft \to {}^\square G.$$

If (d) $E_\square = {}^\square C = 0$, then all three of these sequences can be written:

$$D_\square \to A_\leftarrow \to A_\square \to B_\upharpoonleft \to B_\leftarrow \to {}^\square F \to F_\upharpoonleft \to {}^\square G.$$

4. Some rows and some columns.

In the preceding section we saw what happens when all columns and all but a finite number of rows of a weakly bounded double complex are exact; the corresponding results, of course, hold when all rows and all but finitely many columns are exact.

More generally, what if all but m rows and all but n columns are exact? In this section I shall indicate the sort of behavior that occurs. I omit the proofs; these use the same ideas as in the preceding section, but in general, the situations "before", "at" and "after" each intersection of a nonexact row and a nonexact column require slightly different arguments.

The case $m + n \leq 2$ is covered by Corollary 14 above. The case $m+n = 3$ is covered (up to row-column reversal) by Lemma 18 and

Lemma 20. Suppose we have a weakly bounded double cochain complex with objects A_{rs}, all columns exact but the i^{th}, and all rows exact but the j^{th} and k^{th}, where $j < k$. Then writing $m = i-j$, $n = i-k$, we have an exact sequence

$$\cdots \to A_{k,r+n}{\leftarrow} \;\to A_{i,r}{\uparrow} \;\to A_{j,r+m}{\leftarrow} \;\to A_{k,r+m+1}{\leftarrow} \;\to \cdots .$$

This begins to look as though the general case will behave as nicely as the special cases $m=0$ and $n=0$. But this fails when $m+n = 4$. We saw in Figure 23 what happens when we have 4 not-necessarily-exact rows. Figure 24 shows a double complex with three not-necessarily-exact rows and one not-necessarily-exact column. (The arrows of these rows and column are darkened, and the elements denoted by letters rather than dots.) Figure 25(a) shows the system of "half-long" exact sequences that this double complex leads to. The "ends" look like Figure 23, and go on like this indefinitely, but there is a peculiar "splicing" in the middle. Higher values of m and/or n yield

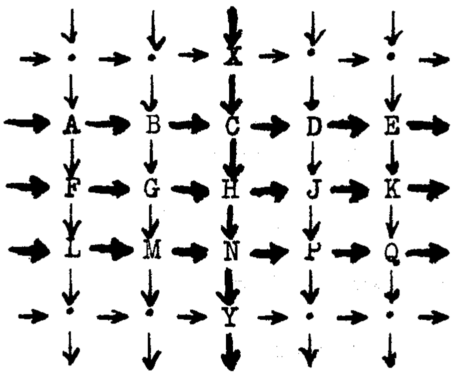


Figure 24.

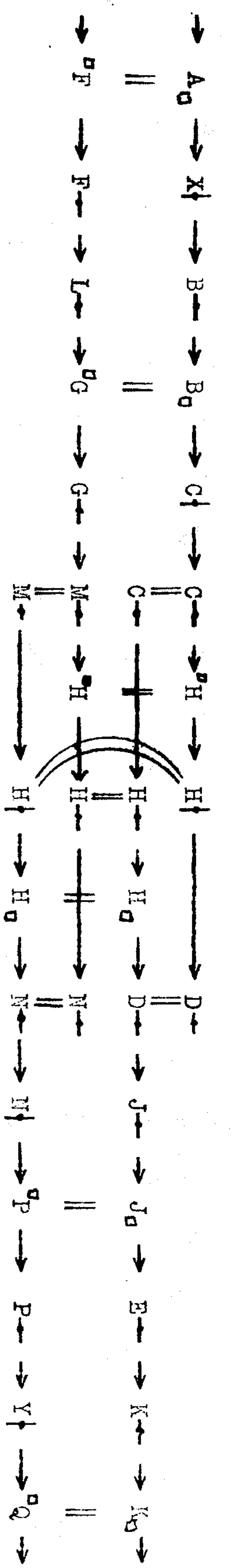


Figure 25(a)

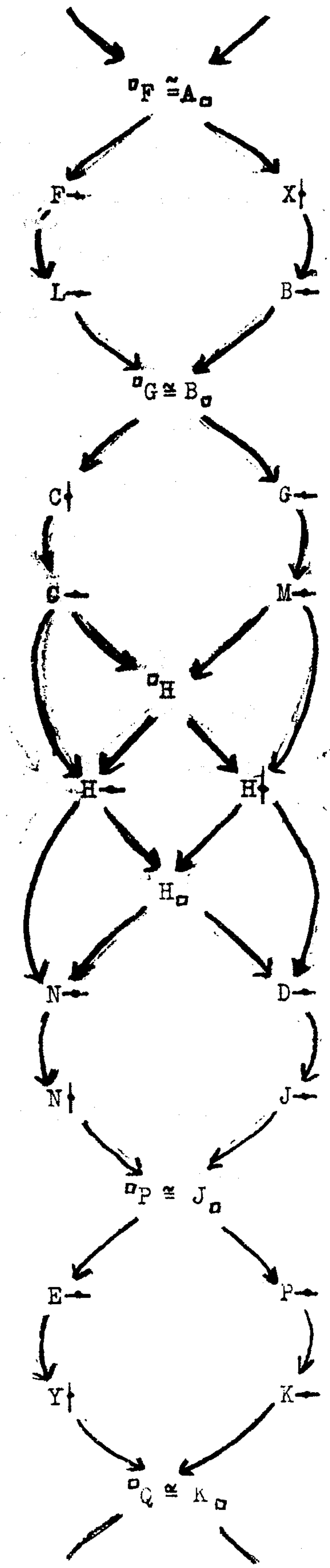


Figure 25(b)

X↕ $\cong$ $^{\square}X$ $\cong$ $^{\square}B$	Y↕ $\cong$ $Y_{\square}$ $\cong$ $P_{\square}$
C↕ $\cong$ $^{\square}C$	N↕ $\cong$ $N_{\square}$
C↔ $\cong$ $C_{\square}$	N↔ $\cong$ $^{\square}N$
D↔ $\cong$ $D_{\square}$ $\cong$ $^{\square}J$	M↔ $\cong$ $^{\square}M$ $\cong$ $G_{\square}$
E↔ $\cong$ $E_{\square}$ $\cong$ $^{\square}K$	L↔ $\cong$ $^{\square}L$ $\cong$ $F_{\square}$
$^{\square}P$ $\cong$ $J_{\square}$	$^{\square}G$ $\cong$ $B_{\square}$
$^{\square}Q$ $\cong$ $K_{\square}$	$^{\square}F$ $\cong$ $A_{\square}$

Figure 26.

systems with more "splicings"; essentially, there is one for each object of the given double complex which lies at the interesection of an inexact row and an inexact column, and does not have either to its upper right or lower left a region where all donors and receptors are zero (due to exactness and weak boundedness). The one splicing in Figure 25(a) comes from the unique object H of Figure 24 with these properties.

Figure 25(b) shows the same exact sequences as Figure 25(a), but arranged to show more clearly how the sequences interlock. The exact sequences are those chains of arrows which can be followed "smoothly" down the diagram. (Figure 26 lists for the benefit of the reader who wishes to check the exactness of the sequences of Figure 25 the isomorphisms of objects that he should first verify. He should then, reversing the final step in the derivation of these diagrams, make the substitutions indicated by the 3-term isomorphisms, e.g., $^{\circ}B$ for $X|$, throughout the diagram.)

The interpolation of some exact rows between the three inexact rows of Figure 24 will not affect the resulting system of exact sequences, Figure 25, except by a shift of indices.

Figures 27 and 28 show the case $m = n = 2$. Here there are two "bad" objects, C and L, and hence two "splicings". If exact rows or columns are introduced between the given inexact ones, the splicings move farther apart, with a longer "normal" stretch between them.

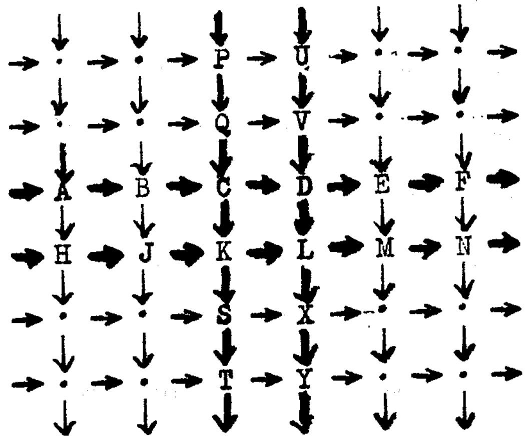


Figure 27

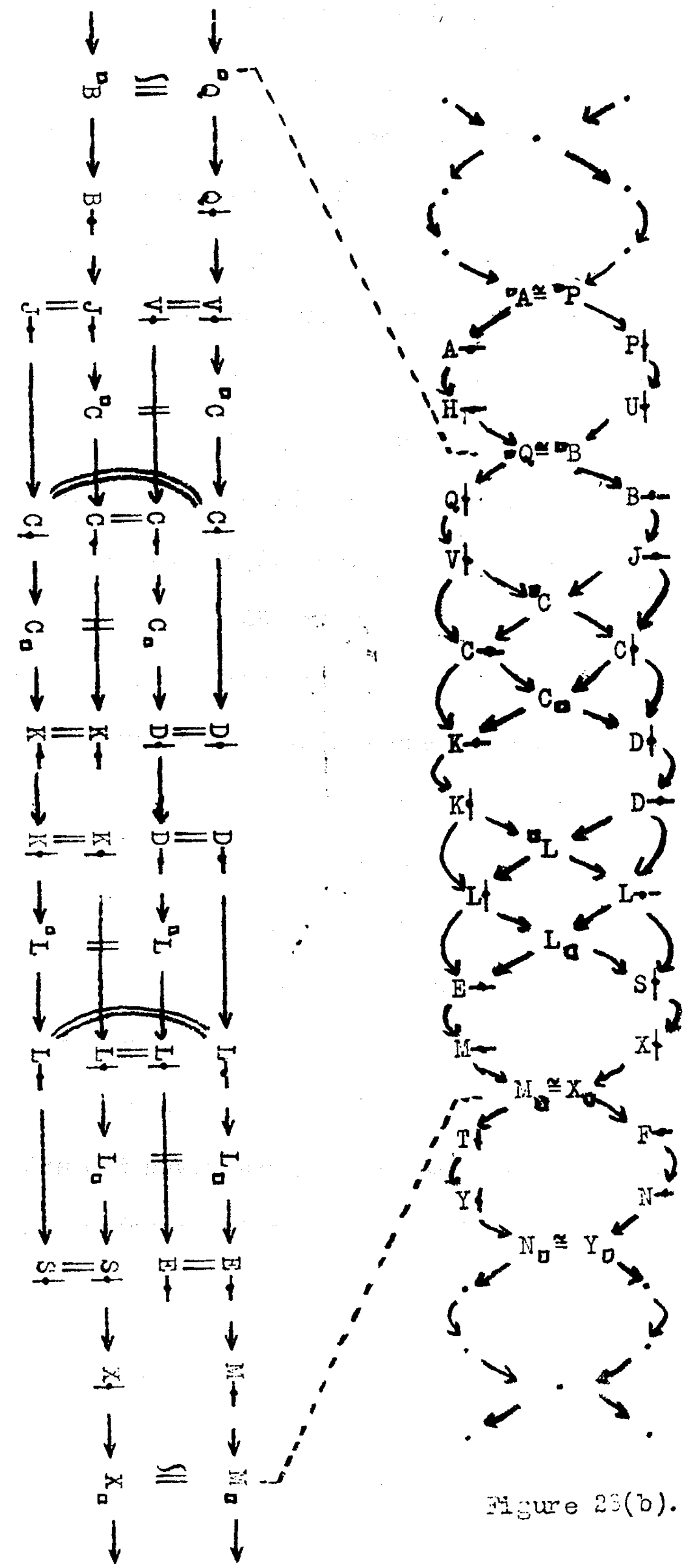


Figure 28(a.)

Figure 28(b).

5. Total homology.

It is probably foolhardy, at very least, for someone who does not know spectral sequences to attempty to say something about the total homology of a double complex. However, I shall show here some connections between that subject and the constructions $^{\sigma}A$ and A_{σ} we have been working with.

Let us be given a double cochain complex with objects $A_{i,r}$ $(i, r \in \mathbb{Z})$, vertical arrows denoted $\delta_1 : A_{i,r} \to A_{i+1,r}$ and horizontal arrows $\delta_2 : A_{i,r} \to A_{i,r+1}$. Thus, at each object $A_{i,r}$ we have

(5) $$\delta_1 \delta_2 = \delta_2 \delta_1, \quad \delta_1 \delta_1 = 0, \quad \delta_2 \delta_2 = 0.$$

At this point one usually defines the total complex induced by this double complex to have for objects the direct sums $A_n = \oplus_{i+r=n} A_{i,r}$ (assuming them defined in our category $\underset{\sim}{A}$). But we may as well be more general. Let $\underset{\sim}{A}^{\mathbb{Z}}$ denote the abelian category of all $\mathbb{Z}$-tuples $X = (X_i)_{i \in \mathbb{Z}}$ of objects of $\underset{\sim}{A}$, and let $\sum : \underset{\sim}{A}^{\mathbb{Z}} \to \underset{\sim}{B}$ denote any exact functor into an abelian category $\underset{\sim}{B}$, commuting with shift; i.e., such that for any objects X_i $(i \in \mathbb{Z})$ of $\underset{\sim}{A}$ we have a functorial isomorphism:

generalization in the literature?

(6) $$\sum_i X_i \cong \sum_i X_{i+1}.$$

For instance, with $\underset{\sim}{A} = \underset{\sim}{B}$ = the category of all R-modules (R any ring) we might take $\sum$ to be (i) direct sum,* or (ii) direct product, or

*In any abelian category $\underset{\sim}{A}$ with countable direct sums (coproducts), the functor $\sum = \oplus_{i \in \mathbb{Z}} : \underset{\sim}{A}^{\mathbb{Z}} \to \underset{\sim}{A}$ is right exact and satisfies (6). For $\underset{\sim}{A}$ the category of all R-modules it is easy to check that this construction is also left exact, and likewise that the direct product construction is right exact. But there are abelian categories whose countable direct sums or products are not exact: In the category $\underset{\sim}{A}$ of torsion abelian groups, arbitrary direct products are given by the torsion subgroup of the direct product as groups ([1] Exercise I.3). If p is any prime, it is easy to see that the direct product in this category of the family of short exact sequences $0 \to Z_{p^i} \to Z_{p^{i+1}} \to Z_p \to 0$ $(i \geq 0)$ is not right exact.

(iii) or (iii') the right- or left-truncated "Laurent sum" operations (generalizing formal Laurent series), $\prod$-$\oplus$ and $\oplus$-$\prod$, defined by $\prod\text{-}\oplus\ A_i = (\prod_{i<0} A_i) \times (\oplus_{i\geq 0} A_i)$ and $\oplus\text{-}\prod A_i = (\oplus_{i<0} A_i) \times (\prod_{i>0} A_i)$ (verify (6)!), or some pathological construction such as (iv) $\sum A_i = \prod A_i / \oplus A_i$.

We now define $A_n = \sum_i A_{i,n-i}$ ($n \in \mathbb{Z}$). The families of maps δ_1 and δ_2 induce maps which we shall denote by the same symbols, $\delta_1, \delta_2: A_n \to A_{n+1}$. (The isomorphism (6) is used in the definition of δ_1!) These will again clearly satisfy (5). Since δ_1 and δ_2 now represent maps which can simultaneously have the same range and domain, we may add and subtract them, and (5) immediately yields

$$(\delta_1+\delta_2)(\delta_1-\delta_2) = 0 = (\delta_1-\delta_2)(\delta_1+\delta_2).$$

Thus, if we define for each n, $\delta = \delta_2 + (-1)^n \delta_1 : A_n \to A_{n+1}$, we get a cochain complex

$$\ldots \xrightarrow{\delta} A_{n-1} \xrightarrow{\delta} A_n \xrightarrow{\delta} A_{n+1} \xrightarrow{\delta} \ldots$$

which we shall call the total complex (with respect to the functor $\sum$) of our given double complex. Since the maps δ come from maps going downward and to the right in our original double complex, we shall denote the cohomology objects of the above complex by

(7) $$A_{n\searrow} = \mathrm{Ker}(A_n \xrightarrow{\delta} A_{n+1})/\mathrm{Im}(A_{n-1} \xrightarrow{\delta} A_n)$$

So far, nothing is new. We now bring our donor and receptor objects into the picture.

Let us define

(8) $$A_{n\Box} = \sum A_{i,n-i\,\Box}, \qquad A_n = \sum {}^{\Box}A_{i,n-i}.$$

From the exactness of $\sum$, it follows that

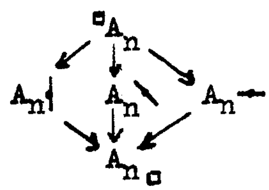
(9)
$$A_{n\Box} = \mathrm{Ker}(\delta_1\delta_2)/(\mathrm{Im}\,\delta_1 + \mathrm{Im}\,\delta_2)$$
$${}^{\Box}A_n = (\mathrm{Ker}\,\delta_1 \cap \mathrm{Ker}\,\delta_2)/\mathrm{Im}(\delta_1\delta_2).$$

(In the "Ker"s in the above equations, $\delta_1\delta_2$ etc. refers, of course, to the map by that name with domain A_n, while in the "Im"s, maps with codomain A_n are meant.)

The identity of A_n now induces intermural maps

$$ {}^{\square}A_n \to A_n^{\diagdown} \to A_{n\square}. $$

(We could also have given definitions and characterizations of $A_n^{|}$ and A_n^{-} analogous to (8) and (9), and gotten a commuting diagram



but we shall have no use for these additional objects and maps here.)

Finally, the two sets of <u>extramural</u> maps constructed from our original double complex in $\mathfrak{Sl}$ yield, for each n, two maps which we shall call $\bar{\delta}_1, \bar{\delta}_2$: $A_{n\square} \to {}^{\square}A_{n+1}$; these are induced in $\underset{\sim}{B}$ by the maps δ_1, δ_2: $A_n \to A_{n+1}$. Let us write:

(10) $$ \bar{\delta} = \bar{\delta}_2 + (-1)^n \bar{\delta}_1 : \quad A_{n\square} \to {}^{\square}A_{n+1}. $$

By (7), we see that the composition of $\bar{\delta}$ with the intermural map

$$ A_{n-1\,\square} \xrightarrow{\ \bar{\delta}\ \ {}^{\square}A_n\ } A_n^{\diagdown} $$

is zero. This says that the two compositions

$$ A_{n-1\,\square} \underset{\bar{\delta}_2}{\overset{\bar{\delta}_1}{\rightrightarrows}} {}^{\square}A_n \longrightarrow A_n^{\diagdown} $$

are equal up to sign. Hence below, we shall only refer to the composition involving $\bar{\delta}_1$; and the same will apply to the composite maps $A_{n\square} \longrightarrow A_{n+1}^{\diagdown}$.

<u>Lemma 20.</u> For each n, the sequence of intermural and extramural maps and their compositions

(11) $$ A_{n-1\,\square} \xrightarrow{\ \bar{\delta}_1\ \ {}^{\square}A_n\ } A_n^{\diagdown} \longrightarrow A_{n\square} \xrightarrow{\ \bar{\delta}\ } {}^{\square}A_{n+1} \longrightarrow A_{n+1}^{\diagdown} \xrightarrow{\ A_{n+1\,\square}\ \bar{\delta}_1\ } {}^{\square}A_{n+2} $$

is exact.

Proof. We can get this by a trick from Lemma 5. We construct a double complex in $\underset{\sim}{B}$ having objects $B_{i,r} = A_{i+r}$, having δ for horizontal maps, and δ_1 for vertical maps (Figure 29). From (5) and (10) we see that $\delta_1\delta = \delta_1\delta_2 = \delta_2\delta_1 = \delta\,\delta_1$, and that in each object, Im δ_1 + Im δ = Im δ_1 + Im δ_2, Ker $\delta_1 \cap$ Ker δ = Ker $\delta_1 \cap$ Ker δ_2. It easily follows that $B_{i,r}^{\nwarrow} = A_{i+r}^{\nwarrow}$, $B_{i,r\,\square} = A_{i+r\,\square}$, ${}^{\square}B_{i,r} = {}^{\square}A_{i+r}$, and that (11) is just the 6-term exact sequence associated with any horizontal arrow of this double complex.

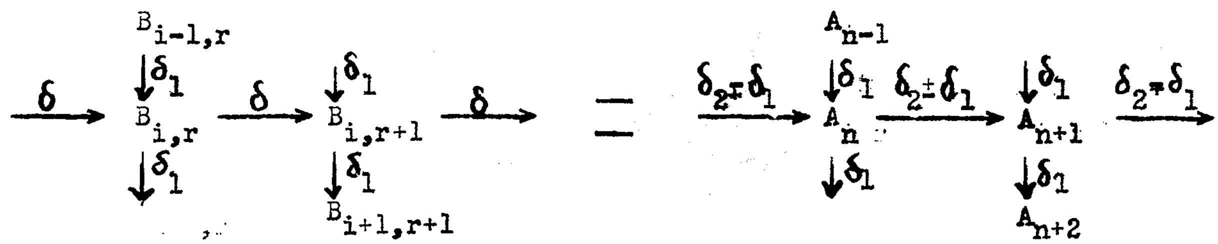


Figure 29.

Now Corollary 6 (or if one prefers, Corollary 16 applied to any row of the auxiliary double complex of the above proof) gives:

Corollary 21. If our original double complex has exact columns, then one has a long exact sequence in $\underset{\sim}{B}$:

$$(12)\quad \cdots \xrightarrow{\bar{\delta}} {}^{\square}A_n \longrightarrow A_n^{\nwarrow} \longrightarrow A_{n\,\square} \xrightarrow{\bar{\delta}} {}^{\square}A_{n+1} \longrightarrow \cdots$$

(Note the curious property of this sequence: that the objects joined by the connecting morphisms $\bar{\delta}$ are isomorphic under a different map, $\bar{\delta}_1$ (Cor.6).)

Corollary 22. If our original double complex has exact rows and columns, and is weakly bounded (e.g., if $A_{i,r} = 0$ whenever i or r is negative), then all total homology objects $A_n^{\nwarrow}$ are zero.

Proof. For all n, ${}^{\square}A_n = \sum {}^{\square}A_{i,n-i} = \sum 0 = 0$, and similarly $A_{n\,\square} = 0$. So by (12), $A_n^{\nwarrow} = 0$.

For the remainder of this section, we shall assume as in Corollary 21 that our given double complex has exact columns, and we will investigate the behavior of the exact sequence (12) for various choices of $\underset{\sim}{A}$ and Σ.

Consider the pair of objects $A_{n\square}$ and ${}^{\square}A_{n+1}$ of that sequence. We have two maps between them, one of which is an isomorphism because of our assumption of exact columns:

(13) $$A_{n\square} \underset{\bar{\delta}_2}{\overset{\bar{\delta}_1 \,\sim}{\rightrightarrows}} {}^{\square}A_{n+1}$$

This diagram comes from a system of objects and maps in $\underset{\sim}{A}$:

(14) $$\cdots \xrightarrow{\bar{\delta}_2} {}^{\square}A_{i+1,n-i} \overset{\bar{\delta}_1}{\cong} A_{i,n-i\square} \xrightarrow{\bar{\delta}_2} {}^{\square}A_{i,n-i+1} \overset{\bar{\delta}_1}{\cong} A_{i-1,n-i+1\square} \xrightarrow{\bar{\delta}_2} \cdots$$

(cf. Figure 21). Identifying the successive isomorphic objects of (13'), we may regard this diagram as a directed system in $\underset{\sim}{A}$. When (13') has a direct limit in $\underset{\sim}{A}$, we shall denote this, by abuse of terminology, $\underrightarrow{\mathrm{Lim}}\, A_{n\square}$, or $\underrightarrow{\mathrm{Lim}}\, {}^{\square}A_{n+1}$.

Now assume that $\underset{\sim}{A}$ has countable coproducts. Then it is well-known that it will also have countable direct limits, and easy to check that the direct limit of (14) can be constructed as the cokernel of the map

(15) $$\bar{\delta}_2 - \bar{\delta}_1 : \oplus A_{i,n-i\square} \longrightarrow \oplus\ {}^{\square}A_{i,n-i+1}.$$

Now suppose that countable coproducts are exact in $\underset{\sim}{A}$, and take $\Sigma = \oplus$. Then (up to a possible change of sign of $\bar{\delta}_1$, which obviously won't change the direct limit of (14)) we see that (15) is just

(16) $$\bar{\delta} : A_{n\square} \longrightarrow {}^{\square}A_{n+1}.$$

In summary: if $\Sigma = \oplus$, then the cokernel of (16) is given by the direct limit of (14).

The kernel of (16) does not have such a natural general description in this situation. But if we take $\underset{\sim}{A}$ to be the category of R-modules (R any ring) this kernel will always be zero! For suppose x is a nonzero element of $A_{n_\square} = \oplus A_{i,n-i_\square}$. Let i be the largest integer such that $0 \neq x_i \in A_{i,n-i_\square}$. One then finds that the ${}^\square A_{i+1,n-i}$ component of $\bar{\delta}(x)$ is nonzero, proving $\bar{\delta}$ injective.

Applying these observations to the exact sequence (12) we get:

Corollary 23. Given a double complex of R-modules (R a ring) with exact columns, if one forms the total complex with respect to "$\oplus$", then the total homology is described by:

$$A_n^{\nwarrow} \cong \varinjlim {}^\square A_n \cong \varinjlim A_{n-1_\square}.$$

Dualizing the observation following (16) we get: If $\sum = \prod$ (for $\underset{\sim}{A}$ a category with exact countable direct products) then the kernel of (16) is the inverse limit of (14). The cokernel is now hard to describe; even for $\underset{\sim}{A}$ the category of R-modules, it is not in general zero. One can get a more complicated result than the analog of Corollary 23, which I will record without detailed argument:

Suppose an element $x \in A_n^{\nwarrow}$ has the property that for any finite subset $I \subseteq \mathbb{Z}$, x can be represented by a cocycle in A_n which has zero component in all $A_{i,n-i}$ $(i \in I)$. Then we will call x a "peekaboo element" (because wherever you look for it, it isn't there!) The set of these elements forms a submodule of $A_n^{\nwarrow}$, which we shall denote $PB(A_n^{\nwarrow})$. Then from (12) we can get a short exact sequence

(17) $$0 \to PB(A_n^{\nwarrow}) \to A_n^{\nwarrow} \to (\varprojlim A_{n_\square} \cong \varprojlim {}^\square A_{n+1}) \to 0.$$

(To see an example in which the left-hand term of (17) is nonzero, the reader may examine the total homology of the double complex shown in Figure 30,

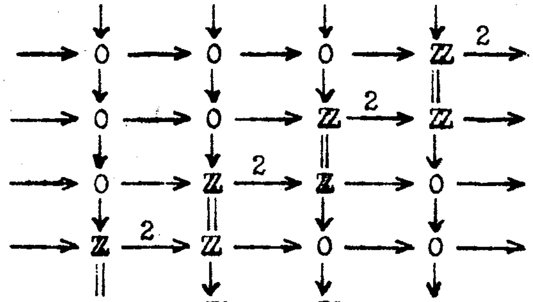


Figure 30.

with respect to $\prod$, at the "upper" nonzero diagonal. Remark: since peekaboo elements can be characterized as those such that <u>any</u> representing cocycle can be <u>approximated</u> arbitrarily closely (in an appropriate sense) by a coboundary, I expect that they would not appear if one took a category $\underset{\sim}{A}$ whose objects had an appropriate <u>completeness</u> property.)

(Incidentally, it is not hard to show, still for $\underset{\sim}{A}$ = R-modules and $\sum = \prod$, that if the maps $\bar{\delta}_2$ of (14) are all surjective, then the cokernel of (16) is zero. In particular, if our double complex has both exact rows and exact columns, and we let $\tilde{A}_{n\square} = {}^{\square}\tilde{A}_{n+1}$ denote the common value up to isomorphism of all the terms (equivalently, the direct <u>or</u> inverse limit) of (14), then from (12) we get: $A_n \cong \tilde{A}_{n\square} \cong {}^{\square}\tilde{A}_{n+1}$.)

We now come to the best behaved — or most trivial — case: $\underset{\sim}{A}$ = R-modules, $\sum$ = the right-truncated Laurent sum functor $\prod - \oplus$. Thus

$$A_n = (\prod_{i<0} A_{i,n-i}) \times (\oplus_{i\geq 0} A_{i,n-i}).$$

Note that in $\bar{\delta} = \bar{\delta}_2 \pm \bar{\delta}_1$, the two terms $\bar{\delta}_1$ and $\bar{\delta}_2$ are each homogeneous, of distinct degrees, with respect to their effects on the first subscripts of $A_{i,n-i}$; and that the summand of higher degree, $\pm\bar{\delta}_1$, is invertible. It follows that $\bar{\delta}$ will be invertible! Indeed, if we write $\bar{\delta} = (1+\varepsilon)(\pm\bar{\delta}_1)$, then $\varepsilon = \pm\bar{\delta}_2\bar{\delta}_1^{-1}$ is homogeneous of degree -1 in these subscripts, and we see that the formal

expression
$$\bar{\delta}^{-1} = \pm\bar{\delta}_1^{-1}(1 - \varepsilon + \varepsilon^2 - \dots)$$

will converge on our ... Laurent sum modules. Corollary 21 now immediately gives:

Corollary 24. A double cochain complex of R-modules with exact columns has trivial total cohomology with respect to the right truncated Laurent sum functor $\prod$-$\oplus$.

With respect to the left truncated Laurent sum operation $\prod$-$\oplus$, δ has kernel and cokernel whose description in terms of the directed system (13) is much more difficult and which I will not try to characterize. (Of course, if we assume rows rather than columns exact in our double complex of R-modules, the behaviors of left and right truncated Laurent sums are reversed.)

It is also easy to show:

Corollary 25. Let $\mathbb{A}$ be any abelian category with exact countable direct products or coproducts (as required), and let us be given a weakly bounded cochain complex with exact columns. Then if we form the total complex with respect to any of the functors $\oplus$, $\prod$, $\oplus$-$\prod$, $\prod$-$\oplus$, the induced maps $\delta: A_n \to {}^{\circ}A_{n+1}$ (see (10)) will be isomorphisms, and hence the total cohomology will be 0.

I have not investigated any other functors $\sum$. A generalization of example (iv) on p.24 is the reduced direct product with respect to any translation-invariant filter on $\mathbb{Z}$.

6. Remarks.

6.1. A more sophisticated formulation for the basic results of this paper (Definitions Lemmas and Corollaries 1-5, 17, 20, 21) would refer to a single object A of an abelian category (possibly graded) with two commuting (or better, anticommuting) square-zero endomorphisms δ_1 and δ_2. In particular, in the formulation of Corollary 17, the vertical exactness assumption would simply take the form $\mathrm{Im}(\delta_1) = \mathrm{Ker}(\delta_1)$, and the diagram obtained, Figure 21, would reduce to an exact couple:

$$\begin{array}{ccc} A_{\square} & \longrightarrow & {}^{\square}A \cong A_{\square} \\ & \nwarrow \quad \swarrow & \\ & A^{\leftarrow} & \end{array} .$$

6.2. Analogous to the one homology object one can associate with each object of a single complex, and the four objects we have associated with each object of a double complex, one may associate to every object A of a triple complex 18 "homological objects"! These are shown schematically in Figure 31. Each of the diagrams on the left side of that figure shows an object A of a double complex, the seven other objects, forming with A the vertices of a cube (a cell of the complex), which have possibly nontrivial maps into A, and the seven objects with possibly nontrivial maps of A into them. Certain vertices of each of these cubes are marked with black dots. The homological object we are defining in each case is

$$\frac{\bigcap \text{(kernels of maps of } A \text{ into the "lower" marked objects)}}{\sum \text{(images of maps into } A \text{ from the "upper" marked objects)}} . \tag{18}$$

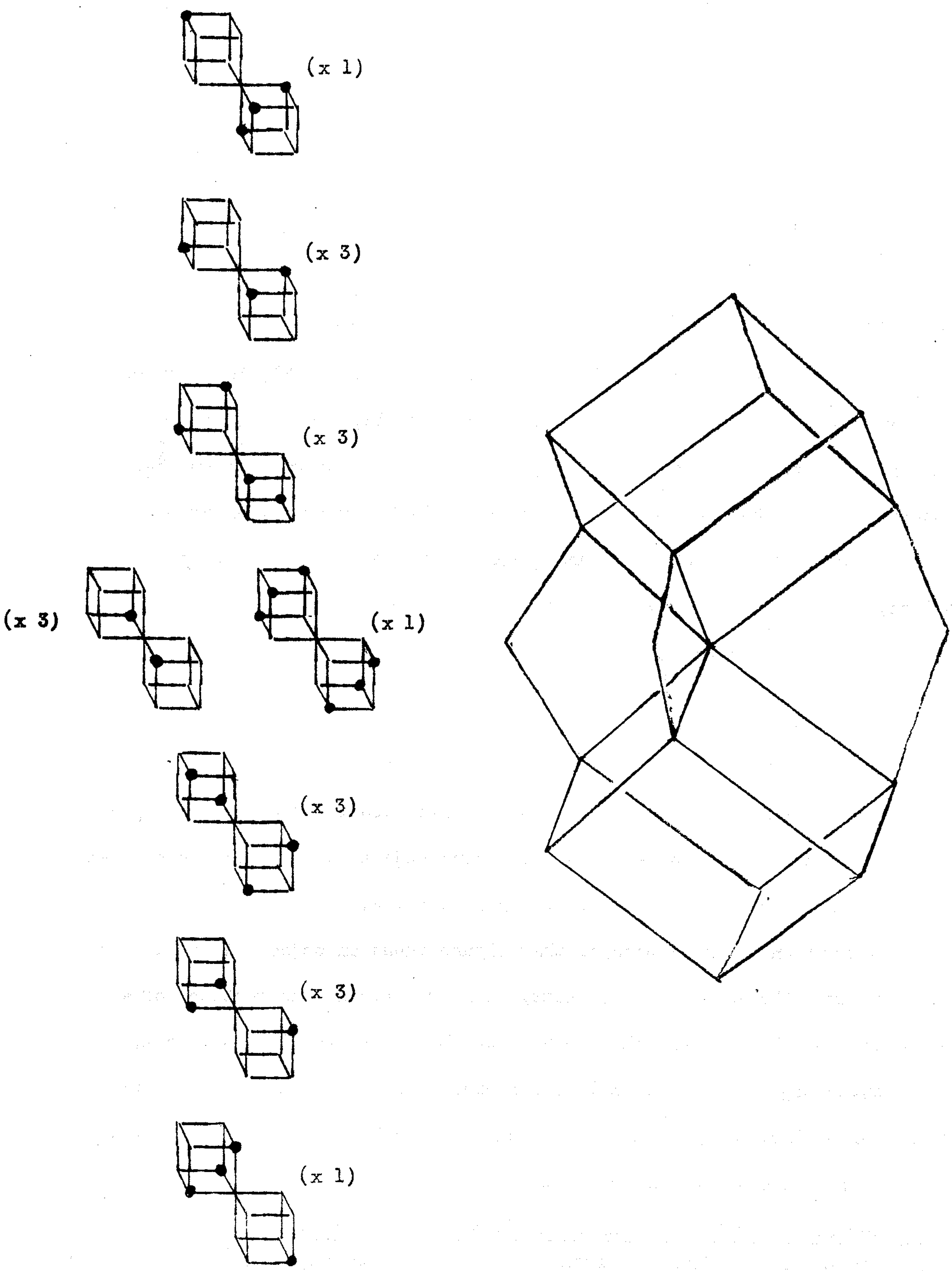


Figure 31.

(When several such objects differ only by a permutation of the three coordinates, I have shown only one representative. The parenthetical "(x 1)" or "(x 3)" thus indicate the number of diagrams that each diagram shown stands for.)

These eighteen constructions are distinguished among the larger number of formally possible ones by the property that the denominator of (18) is in each case the largest sum of images that must lie in the numerator, and the numerator is the smallest intersection-of-kernels submodule (the intersection of the largest irredundant family of kernels) necessarily containing the denominator. (One may look at these eighteen pairs as arising from a Galois connection between vertices of the cell "above" A and vertices of the cell "below" A.) This criterion seems to give the constructions of homological interest. E.g., these are precisely the constructions of the form (18) which are zero for all bounded (almost all objects zero) triple complexes exact in all directions.

The diagram at the right side of Figure 31 shows the intermural maps among these 18 objects. (It forms a lattice because of the properties of Galois connections.) Figure 32, showing the corresponding pictures for simple and double complexes, is given for comparison.

It would be interesting to see what exact sequences or diagrams may link the objects defined in Figure 31. Whether such results would be of any use is another question!

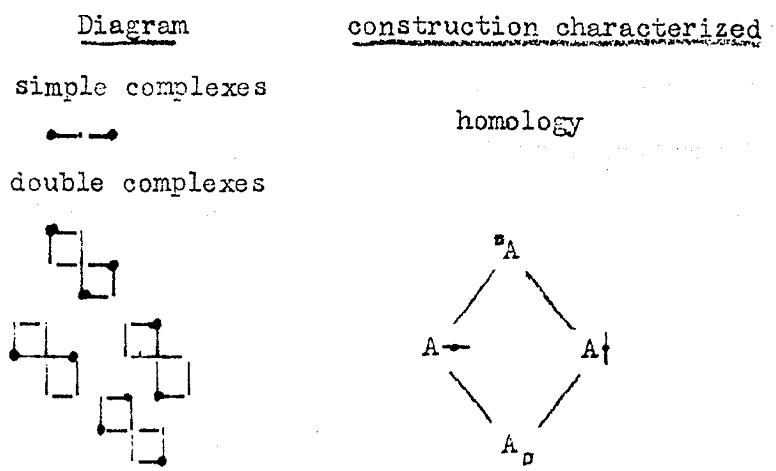


Figure 32.

<u>6.3.</u> J. Lambek [3] (cf. [5] p.96) associates to any commuting square, as in Figure 33, two objects which he call the kernel ratio and the image ratio of

$$\begin{array}{ccc} P & \xrightarrow{a} & R \\ b\downarrow & & \downarrow c \\ Q & \xrightarrow[d]{} & S \end{array} \qquad f = ca = db$$

Figure 33.

the square. To bring out the analogy with the definitions of this paper, let me call them

$$P_* = \mathrm{Ker}\ f / (\mathrm{Ker}\ a + \mathrm{Ker}\ b)$$

$$^*S = (\mathrm{Im}\ c \cap \mathrm{Im}\ d) / \mathrm{Im}\ f.$$

Note that if the given square is embedded in a double complex which is vertically <u>and</u> horizontally exact at P, respectively at S, then one has $P_* = P_{\square}$, respectively $^*S = {}^{\square}S$.

Any commuting diagram as in Figure 34 with exact rows can be extended, by

$$\begin{array}{ccccc} P & \longrightarrow & R & \longrightarrow & T \\ \downarrow & & \downarrow c & & \downarrow \\ Q & \longrightarrow & S & \longrightarrow & U \end{array}$$

Figure 34.

putting in the kernel and cokernel of c, to a double complex exact in both directions at R and S. Hence Corollary 6, applied to c, gives us

$$R_* \cong R_{\square} \cong {}^{\square}S \cong {}^*S.$$

This isomorphism $R_* \cong {}^*S$ is proved by Lambek [3] and used to get various other results, more or less as I use Corollary 6 in §2 above. The constructions $(\ )_*$ and $^*(\ )$ have the advantage of being definable with reference to a smaller diagram than my $(\ )_{\square}$ and $^{\square}(\ )$. They share with these the property of vanishing in any <u>finite</u> <u>doubly</u> <u>exact</u> double complex. But they have the disadvantage that one doesn't seem to be able to do anything with them without exactness assumptions. As we have seen, under such assumptions one has the analog of the extramural isomorphism of Corollary 6, but without them one does not have analogs of extramural maps, or Lemma 5.

Lambek proves his result for not-necessarily abelian groups, though he applies it in abelian situations. (Leicht [4] gives a category-theoretic version of Lambek's result.) However, note that exactness of the top row of Figure 34 is necessary to conclude that $^*S$ will be a group, i.e., that the denominator in the definition is a normal subgroup of the numerator. I noticed when I was first working out the Salamander Lemma that some kind of version could be stated for not-necessarily-abelian groups, but we run into even worse difficulties — e.g., in Definition 1, $A_\square$ would have to be simply a pointed set, the quotient of the group Ker q by the left action of the subgroup Im c and the right action of the subgroup Im d. Yet, it certainly would be desirable to have a tool for proving the noncommutative versions of basic diagram-chasing Lemmas, then these hold.

7. Exercises.

7.1. (a) Let $\underset{\sim}{A}$ be an abelian category, let $\underset{\sim f}{A}^{\#}$ denote the abelian category of double complexes in $\underset{\sim}{A}$ (where morphisms are families of maps $f_{i,r}: A_{i,r} \to B_{i,r}$ commuting with δ_1 and δ_2), and let $FX \subseteq Ob(\underset{\sim}{A}^{\#})$ be the class of double complexes with all rows and columns exact, and only finitely many nonzero objects. Let $EX \subseteq FX$ be the class of double complexes of the simple form shown in Figure 35 ($A \in Ob(\underset{\sim}{A})$). Prove: FX is the least subclass

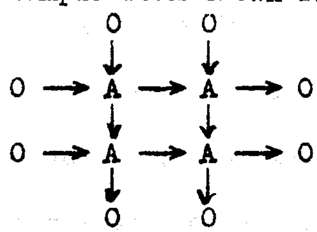


Figure 35

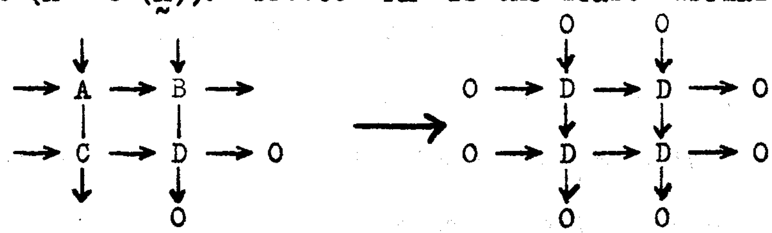


Figure 36

of $Ob(A^{\#})$ containing EX and closed under extensions. (Hint: Given an object of FX, prove that one can map it onto an object of EX as suggested in Figure 36. For surjectivity at A, use $^{\alpha}D = 0$.)

(b) State and prove the analogous result for triple or n-fold complexes. (The complicated homological constructions considered in the preceding section are not needed! $^{\alpha}(\ )$, applied in various planes of the diagram, is still the key!)

(c) Establish the claim (p. 33) that all the constructions indicated in Figure 31 give zero at all objects of a bounded exact triple complex. (Hint: show they give zero on the building-block objects used in part (b).)

(a) Show that from any diagram as shown in Figure 37, in which the short vertical sequences are exact, and the rows $\to A \to \ldots \to H \to$, $\to J \to \ldots \to P \to$, and $\to R \to \ldots \to W \to$ are complexes, one can form a long exact sequence of homologies:

$$\ldots \to C^{\leftarrow} \to L^{\leftarrow} \to T^{\leftarrow} \to D^{\uparrow} \to D^{\leftarrow} \to E^{\leftarrow} \to E^{\uparrow} \to M^{\leftarrow} \to U^{\leftarrow} \to F^{\leftarrow} \to \ldots$$

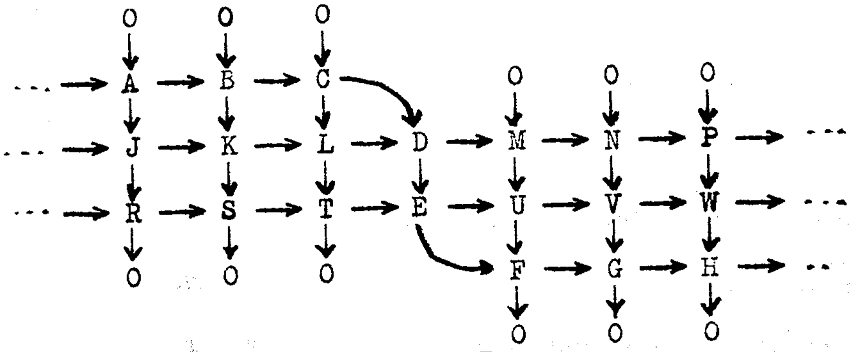


Figure 37.

(b) Given an arrow of a double complex, F → G in Figure 38, obtain the diagram of Figure 39, and apply the preceding result to get a long exact

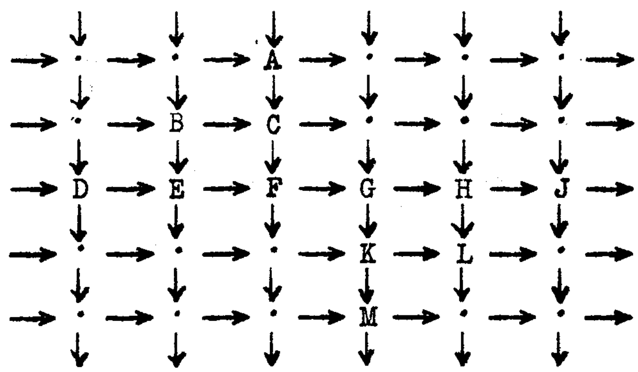


Figure 38.

sequence. Write out the "middle" six terms of that sequence, and then the three on either side of these. Conclusion: we can extend our 6-term "salamander" sequences to long(er) exact sequences if we are willing to define more complicated sorts of auxiliary objects. Do these objects still have the property of being zero on <u>bounded</u> <u>exact</u> double complexes?

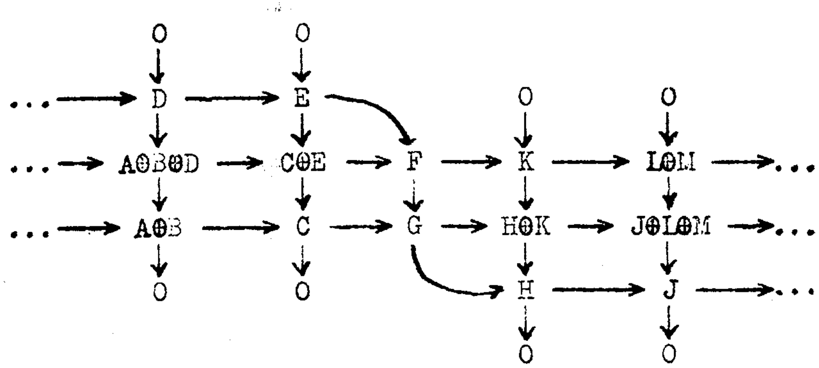


Figure 39.

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